1°) Model Building.

2°) Theory : proving results theorically.

3°) Critical Analysis : when theory fails.

4°) Computation and Simulation.

5°) Real World Implication +

Part I: The model.

1°) The classical Cramér-Lundberg Model.

$$U_t = u + ct - \underbrace{\sum_{i=1}^{N(t)} X_i}_{\text{compound Poisson Process.}}$$

Parameters:

U_t: The Surplus amount at time t.

u: the initial capital.

ct: the constant premium rate.

$N(t)$: the number of claim by time t.

X_i : size of claim i.

2°) **Assumptions:**

* $N(t) \sim \text{Poisson}(\lambda)$.

i.e claims are arriving according to Poisson with rate λ.

Why? ⇒ b/c Poisson Process is the only continuous-time process with independent, stationary increment and discrete jumps. Plus memory less property makes calculation tractable.

* X_i are i.i.d with distribution F (common choice for F:
  * exponential
  * Pareto
  * Gamma, log normal

  with $E(X_i) = \mu < \infty$

* X_i and $N(t)$ are independent.

i.e: claims are independent of each other.

3°) Expected values:

$E(X_i) = \mu$ and $E(N(t)) = \lambda t$

So $E(U_t) = u + ct - \lambda t\mu = u + (c - \lambda\mu)t$.

Which gives the net profit condition: $c > \lambda\mu$.

otherwise if $c \leq \mu\lambda$ then certain ruin.

The Ruin :

$$\tau = \inf \{ t > 0 : U(t) < 0 \}$$

i.e the first time the surplus becomes negative
(if it never hit negative then $\tau = \infty$)

* The ultimate Ruin probability is :

$$\psi(u) = \mathbb{P}(\tau < \infty \mid U(0) = u)$$ $\Rightarrow$ { Probability of ruin with a starting capital u.

So
$\lim_{u \to \infty} \psi(u) = 0$ i.e as you increase the capital then you have a less probability of attaining ruin.

$\lim_{u \to \infty} \psi(u) = 1$ if $c \leq \lambda \mu$.

Task : ~~How~~
How fast does $\psi(u) \to 0$ when $u \to \infty$?
What role do parameters play (λ, μ, c, F)

$\Rightarrow$ The Lundberg's Inequality :

Assuming the net - profit condition holds : i.e $c > \lambda \mu$

And suppose there exist $R>0$ that satisfy :

$$\lambda E[e^{RX}] = \lambda + cR$$

Then $\psi(u) \leq C e^{-Ru}$

Where C is a constant depending on R.

<u>This means</u> :

- The ruin probability decays exponentially in initial capital
- R (called the adjustment coefficient) : quantifies how safe the company is.
- Higher premium rate $\longrightarrow$ Larger R $\longrightarrow$ Safer. (i.e Light-tail $\Rightarrow$ Exponential safety

$\Rightarrow$ The surplus grows linearly but the claim jumps are rare enough that an exponential tail controls ruin risk.

<u>Proof of the Lundberg's Inequality</u> : (use Martingale and Optimal stopping Theorem)

<u>Given</u> : $U(t) = u + ct - S(t)$ where $S(t) = \sum_{i=1}^{N(t)} X_i$

$c > \lambda\mu$.

$R>0$ satisfying : $\lambda E[e^{RX}] = \lambda + cR$

* Define : $M(t) = e^{RU(t)} \cdot e^{-\lambda t (E[e^{RX}] - 1)}$

this is a Martingale.

* Apply OST at τ : $E(M(\tau)) = M(0) = e^{Ru}$

* At ruin $U(\tau) \leq 0 \Rightarrow e^{RU(\tau)} \leq 1$

Therefore $e^{Ru} \geq \psi(u)$

The previou lundberg's inequality

($\psi(u) \le e^{-Ru}$, where R is solution to $\lambda E[e^{Rx}] = \lambda + cR$)

assumes that $E[e^{Rx}] < \infty$ (is finite for some $R > 0$)

But when $E[e^{Rx}] = \infty$, then the Proof For the lundberg's inequality break. ⇒ We have a heavy-tailed distribution

⇒ No adjustment coefficient exists.

⇒ MGf does not exists

⇒ Tails decays slower than exponentially. (Polynomial or slower)

__ex__ : Pareto distribution, log-Normal, etc.

__Figure__ : exponential vs polynomial decay.

same u, but one is much higher $\psi(u)$

exponential decay dist : Exponential, Gamma, Normal

Polynomial decay dist : Pareto, log-Normal.

__Q__: How much more capital is needed ?

* Property catastrophe :
  - Hurricane Katrina : \$41B in loss (single event)
  - Pareto like distribution

* light tailed : - the premium gradually offset claim outflow
  - Ruin occurs via many shocks accumulating.

* Heavy tailed : Ruin happens bc of one giant claim.
  = catastrophic event.