Insurance Surplus Modeling with Poisson Process and Ruin Probability

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Motivations

- September 2008: AIG needs a $85 Billons bailout? Why?
- Hurricane Katrina: Multiple Insurer went brankrupt

Questions:

When does the classical theory works and when does it fail? How much capital does it cost?

The Cramer-Lundberg Model

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i$$

Where:

U(t) : the Surplus funds the company has at time t

c*t: the constant premium

N(t): is the number of claim by time t.

X_i: the claim size of each claim.

Assumptions:

- N(t) ~ Poisson(λ)
- X_i: are i.i.d with distribution F.
  - Common example distribution F are: Exponantial, Gamma, Pareto, Normal, …
- $E[X_i] = \mu$ and $\mu < \infty$ i.e the mean is finite
- The net profit condition is met: (Taken from the Expected value of U(t) $E[U(t)]$)

  i.e $c > \lambda\mu$

The Ruin Probability

Time of Ruin:

- $\tau = \inf\{t > 0 : U(t) \leq 0\}$ (i.e the first time the surplus becomes negative)

If U(t) never hits negative, then $\tau = \infty$

The ruin probability

$$\psi(u) = P(\tau < \infty \,|U(0) = u)$$

Which is the probability of ruin with a starting capital of u.

Intuitvely:

- $\lim_{u \to \infty} \psi(u) = 0$
- $\lim_{u \to \infty} \psi(u) = 1$ if $c \leq \lambda\mu$

Lundberg's Inequality

Assumming the net profit condition hols, i.e. $c > \lambda\mu$

Suppose there exists R>0 that stisfy: $\boldsymbol{\lambda \cdot E[e^{RX}] = \lambda + cR}$

Then $\boldsymbol{\psi(u) \leq C \cdot e^{Ru}}$ **,** where **C** is some constant depending on R

We call R the adjustment coefficient

R quantifies how safe the company is.

The inequality means:

The ruin probability decays exponentially in initial capital

Higher premium rate -> Larger R -> Safer

Proof of the Lundberg’s inequality

Use Martingale and Optional Stopping Theorem

Given:

$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i$

$c > \lambda\mu$

R>0 that stisfy: $\boldsymbol{\lambda \cdot E[e^{RX}] = \lambda + cR}$

Define: $M(t) = \boldsymbol{e^{RU(t)} * e^{-\lambda t(E[e^{RX}]-1)}}$

Apply OST at τ : $E[M(\tau)] = M(0) = \boldsymbol{e^{Ru}}$

At Ruin: $U(\tau) \leq 0 \Rightarrow \boldsymbol{e^{R\,U(\tau)} \leq 1}$

conclude $\boldsymbol{\psi(u) \leq C \cdot e^{Ru}}$

Problem with Lundberg’s Inequality

$\boldsymbol{\psi(u)} \leq \boldsymbol{C} \cdot \boldsymbol{e^{Ru}}$ where R> 0 and is a solution to $\boldsymbol{\lambda} \cdot \boldsymbol{E}[\boldsymbol{e^{RX}}] = \boldsymbol{\lambda} + \boldsymbol{cR}$

Assumes that $\boldsymbol{E}[\boldsymbol{e^{RX}}]$ is finite for some R > 0 (i.e $\boldsymbol{E}[\boldsymbol{e^{RX}}] < \infty$)

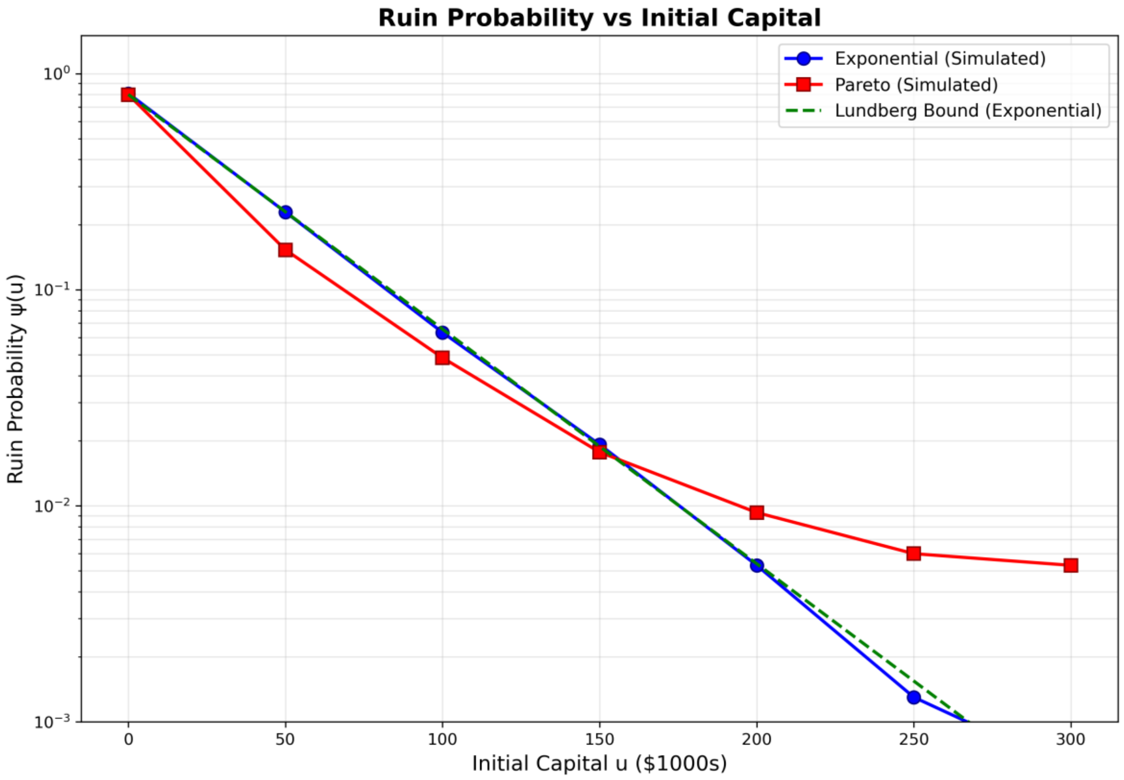
When $\boldsymbol{E}[\boldsymbol{e^{RX}}] = \infty$ then we can’t prove the inequality.

This means:

- We have a heavy-tailed distribution
- No adjustment Coefficient R exists.
- MGF does not exists
- Tails decay slower than exponentially. (Polynomial or slower)

Example of when heavy tails occur: claims are Pareto distributed, or Log-Normal

Exponential vs Polynomial decays



Simulation Algorithm: Event-driven Monte Carlo

For each simulation run:

1. Initialize: $U = u$, $t = 0$

2. Generate next claim arrival time:

   $\tau \sim \text{Exponential}(\lambda)$

3. Accumulate premiums until claim:

   $U \leftarrow U + c \cdot \tau$

4. Process claim:

   Draw $X \sim F$ (claim size)

   $U \leftarrow U - X$

5 .Check ruin:

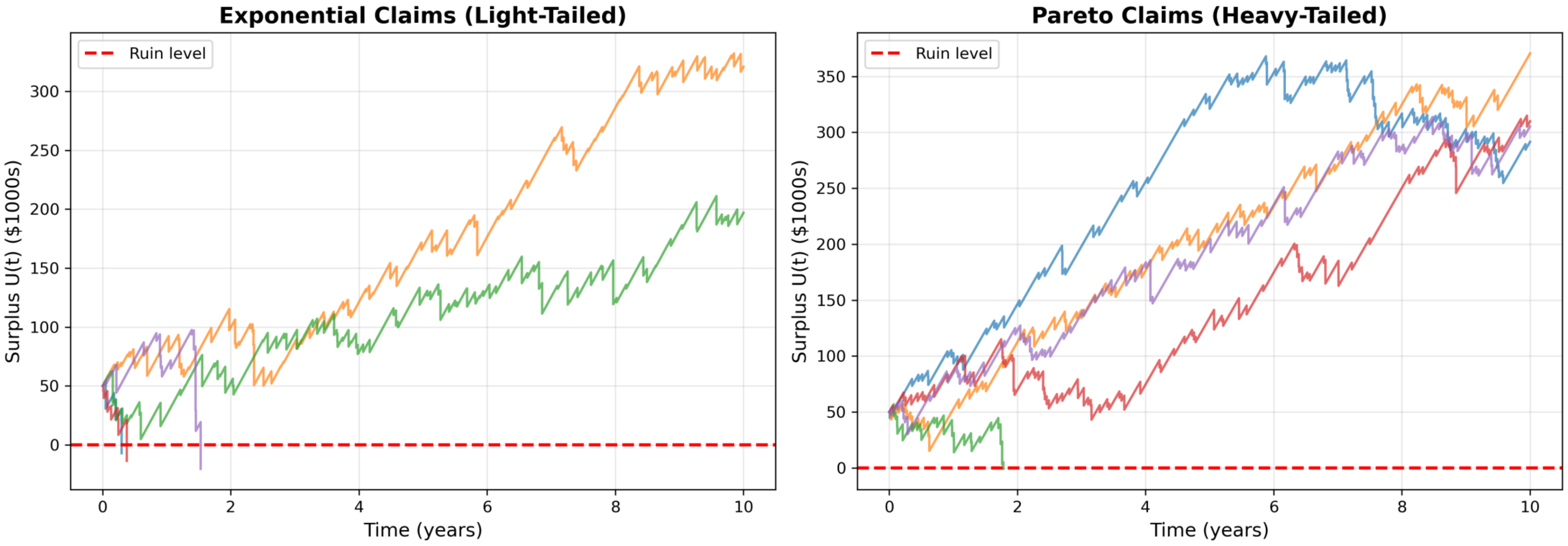
   If $U < 0 \rightarrow$ STOP (ruin occurred)

   Otherwise $\rightarrow$ Return to step 2

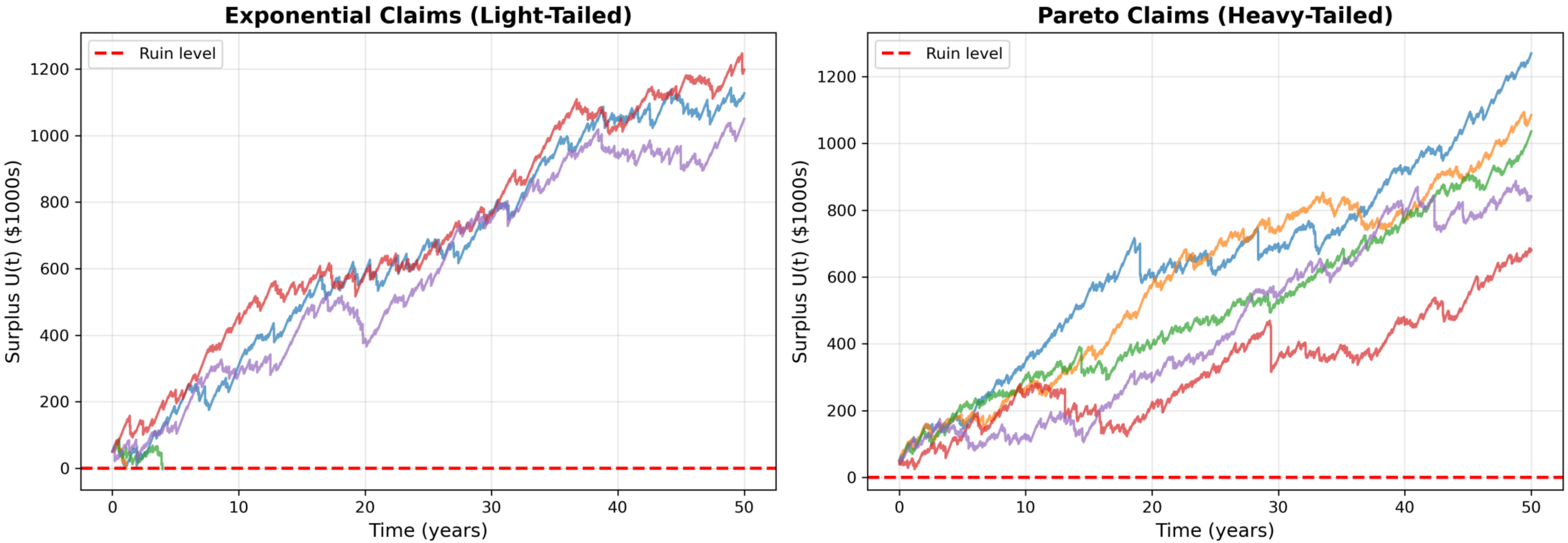
6. Repeat until $t \geq T$ (time horizon)

Estimate: $\psi(u) \approx$ (# ruined runs) / (total runs)

Sample Paths: Exponential vs Pareto Claims



Sample Paths: Exponential vs Pareto



Simulation Results:

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SIMULATION RESULTS SUMMARY
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Initial        Exponential    Pareto         Ratio
Capital        Ruin Prob      Ruin Prob      (P/E)
----------------------------------------------------------------------
$0             81.0%          79.8%          1.0x
$50,000        22.8%          15.2%          0.7x
$100,000       6.3%           4.8%           0.8x
$150,000       1.9%           1.8%           0.9x
$200,000       0.53%          0.93%          1.8x
$250,000       0.13%          0.60%          4.6x
$300,000       0.06%          0.53%          8.8x

Key Observations:
• Pattern: Ratio increases with capital
• At $100K: Pareto is 0.8x times riskier
• At $300K: Pareto is 8.8x times riskier
• Exponential decay (blue) vs. polynomial decay (red)

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Capital Requirements (1% ruin target):
  Exponential: $183,094
  Pareto: $195,882
  Ratio: 1.07x
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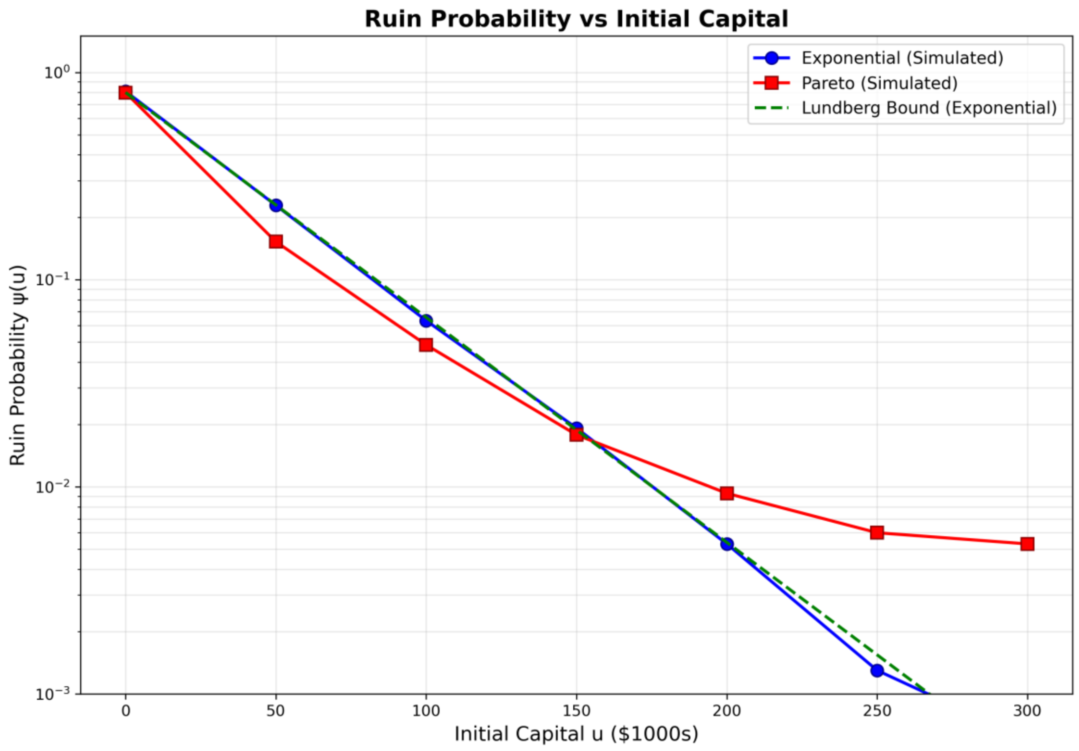

Observation / Interpretation:

Exponential distribution, Validated the classical theory:
Exponential decay

Simulation matches Lundberg bound

Pareto Distribution:

Shows polynomial decay

No exponential bound exists

Capital requirement: Pareto needs higher requirement for 1% ruin probability.

Limitations

Simple Assumption;

- Pure Compound Process: No return on surplus,

- No risk Management: like re-insurance

- single tail type: Most time both light then sudden catastrophic heavy tails

- Parameter Certainty: in reality limited data makes α, λ, c hard to determine

Future works:

- Regime Switching: Light tail in normal then transition to heavy tails in time of crisis
- Optimal Reinsurance: transfer heavy tail risks, capital injections, etc
- Multi-line insurance: Multiple correlated portfolio

References

Ross, S. M. (2014). *Introduction to Probability Models* (11th ed.). Academic Press.
Tomita, Y., Miyagi, K., & Watanabe, K. (2022). Ruin probabilities under mixed Poisson processes.

Aurzada, F. & Buck, M. (2020). Ruin probabilities with insurance and financial risks having temporarily negative capital.

Barrera, M., Rojas, F., & Villaseñor, J. A. (2020).
On the ruin probability of a generalized Cramér–Lundberg model driven by mixed Poisson processes.

Asmussen, S., & Albrecher, H. (2010). *Ruin Probabilities* (2nd ed.). World Scientific.Lundberg, F. (1903). *Approximations of the Probability Function*. Almqvist & Wiksell.
Cramér, H. (1930). *On Collective Risk Theory*. Skandia Jubilee Volume.